Cyclotomic Aperiodic Substitution Tilings

with Dense Tile Orientations

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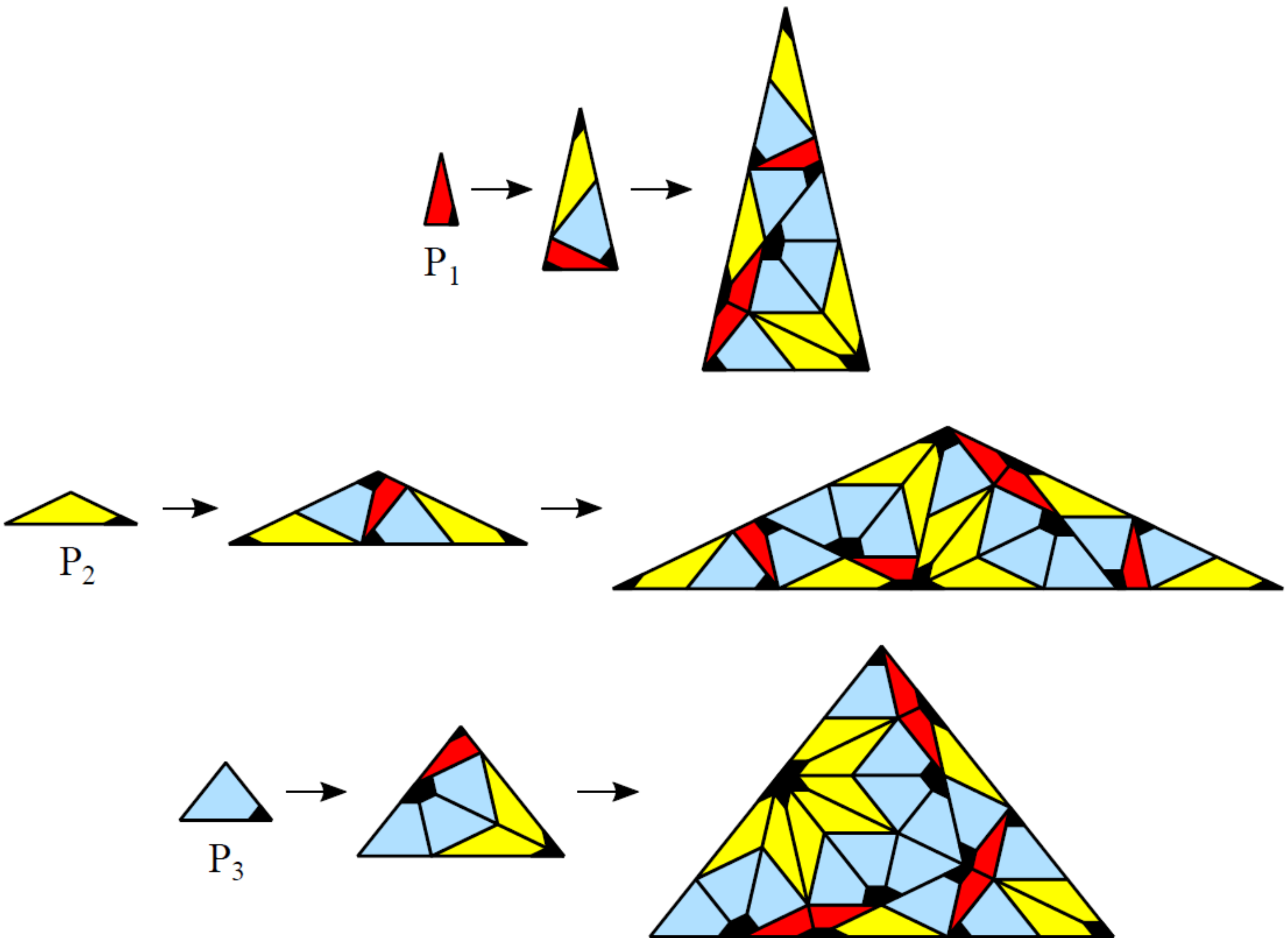
Definitions ...

- A tiling is a countable set of tiles, which is a covering as well as a packing of the plane → No gaps, no overlapping.
- A tiling is cyclotomic if all vertices are supported by the cyclotomic field $\mathbb{Q}\left(\zeta_{2n}\right)$.
- A tiling is called aperiodic if no translation maps the tiling to itself.
- Substitution means, that a set of proto tiles is expanded with a linear map and dissected into copies of proto tiles in original size, by a substitution rule.

Example for a set of Substitution Rules … CAST Δ7-1.1.1



Example: Properties of … CAST Δ7-1.1.1

Substitiution matrix:

$$M = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Left and right eigenvector are related to relative areas and frequencies of the tiles:

$$x_A = x_f = \begin{pmatrix} \mu_{7,3} \\ \mu_{7,2} \\ \mu_{7,1} \end{pmatrix} = \frac{1}{\sin\left(\frac{\pi}{7}\right)} \begin{pmatrix} \sin\left(\frac{3\pi}{7}\right) \\ \sin\left(\frac{2\pi}{7}\right) \\ \sin\left(\frac{\pi}{7}\right) \end{pmatrix}$$

Eigenvalue is a real number and an algebraic integer of the cyclotomic field:

$$\lambda = \mu_{7,3} + \mu_{7,2} + \mu_{7,1} = \mu_{7,3}^2$$

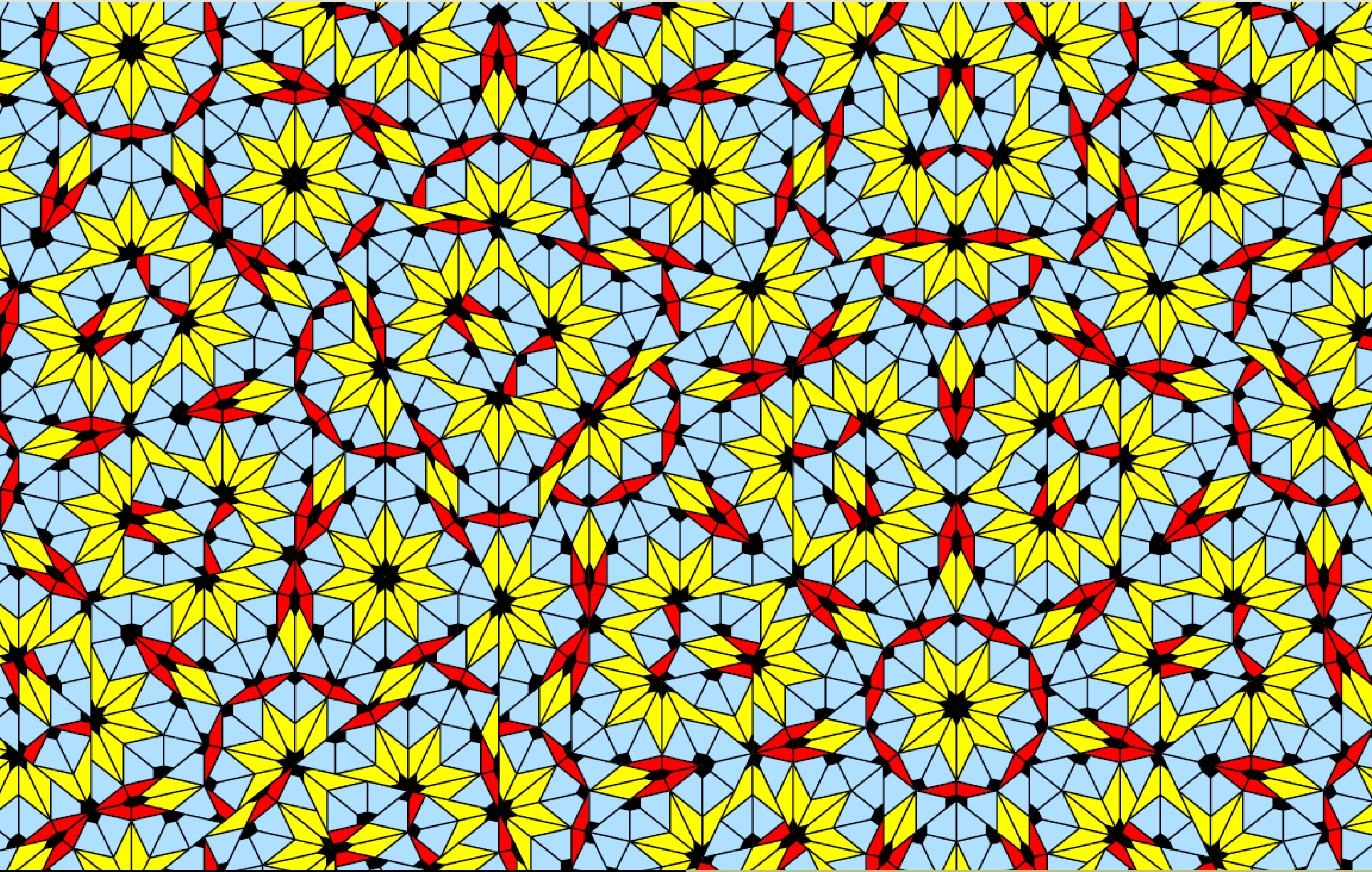
$$\lambda = \eta\overline{\eta} \in \mathbb{Z}\left[\mu_n\right] = \mathbb{Z}\left[\zeta_{2n} + \overline{\zeta_{2n}}\right]$$

Inflation multiplier is also a real number and an algebraic integer of the cyclotomic field:

$$\eta = \mu_{7,3} \in \mathbb{Z}\left[\mu_n\right] = \mathbb{Z}\left[\zeta_{2n} + \overline{\zeta_{2n}}\right]$$

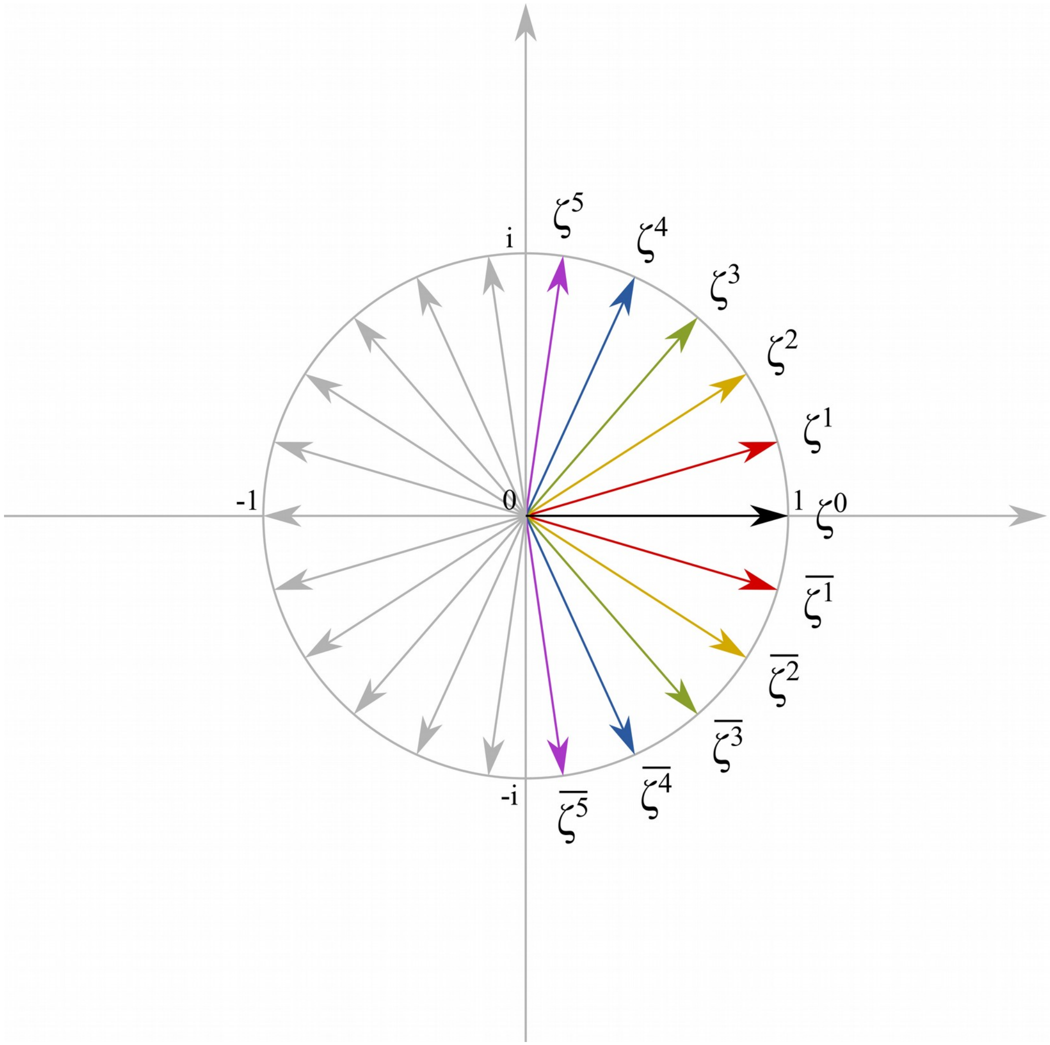
All proto tiles appear in a finite number of orientations. (Finite tile orientations)

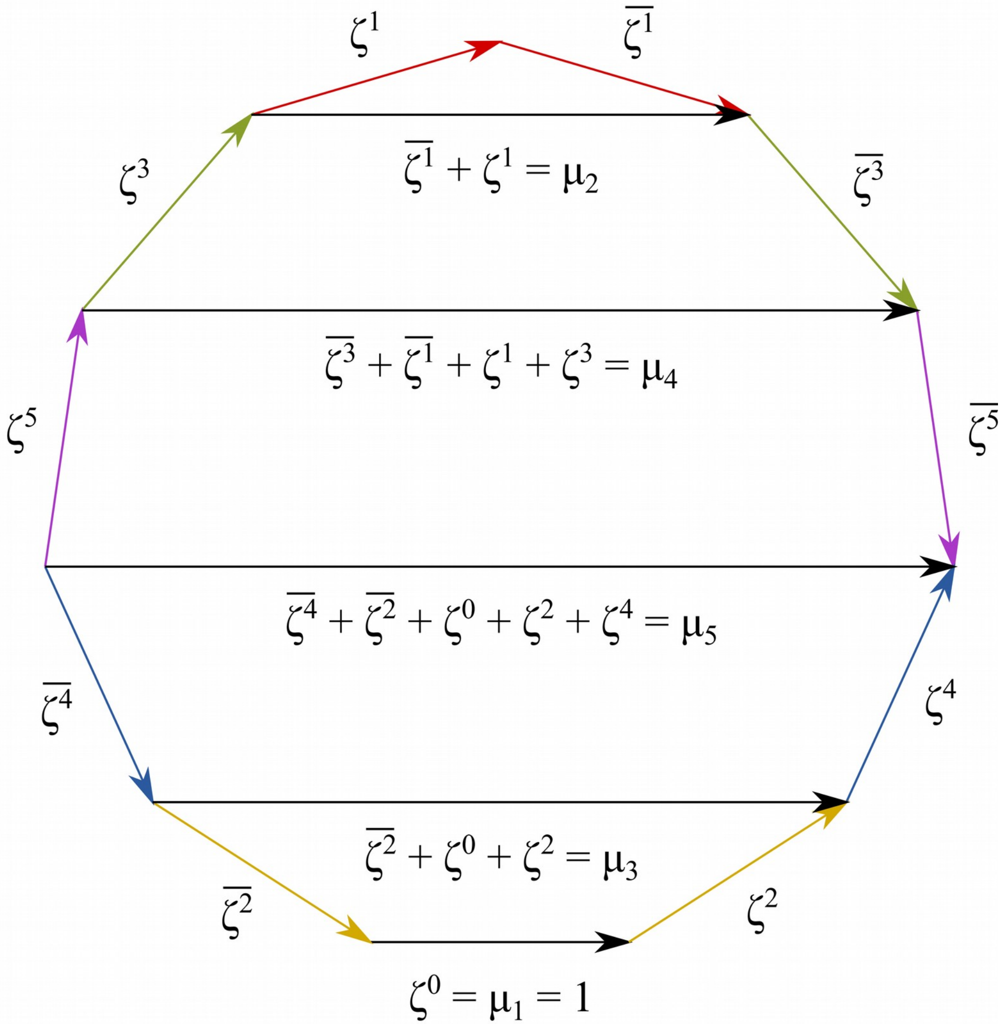
Example for a Substitution Tiling ... CAST Δ7-1.1.1



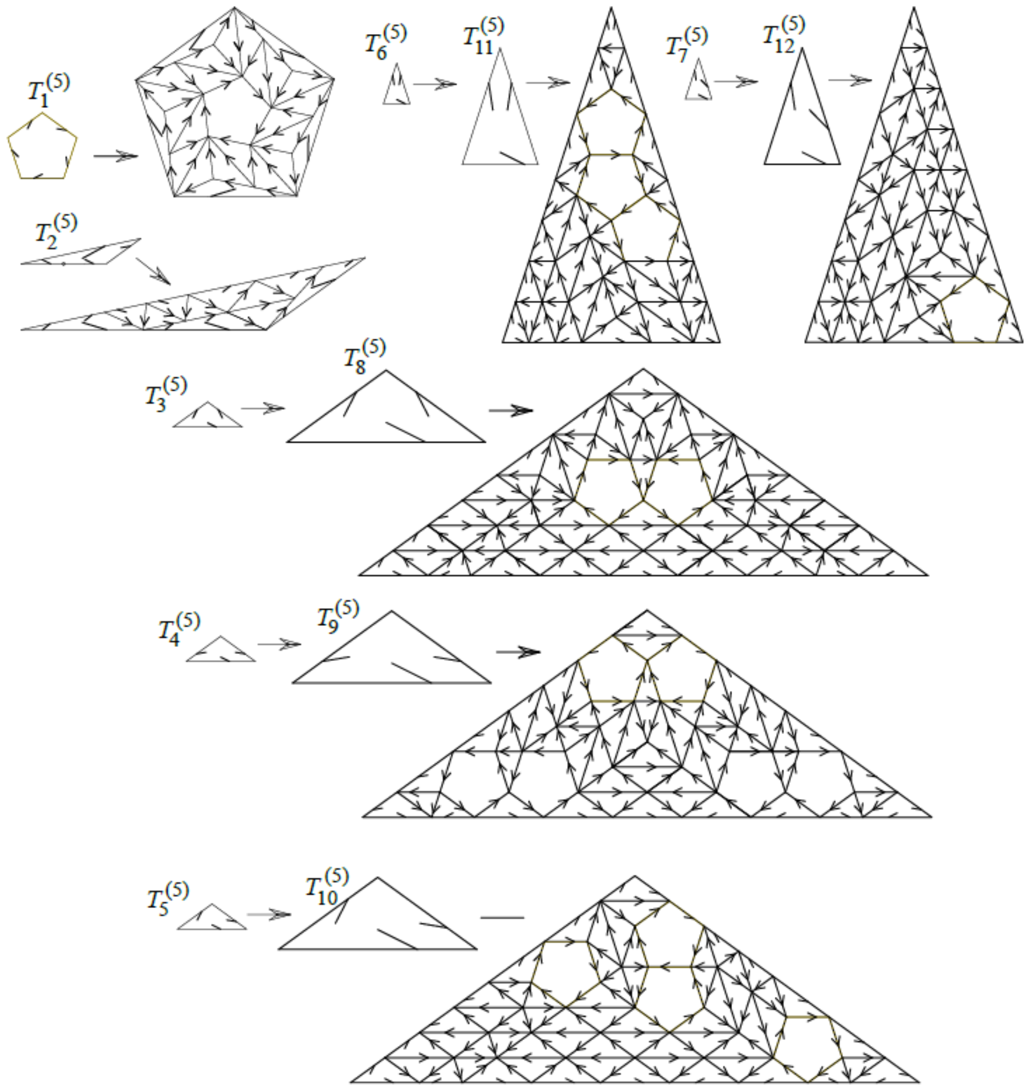
Roots of Unity vs. Diagonals of the n-gon

$$\mathbb{Z}\left[\mu_n\right] = \mathbb{Z}\left[\zeta_{2n} + \overline{\zeta_{2n}}\right]$$

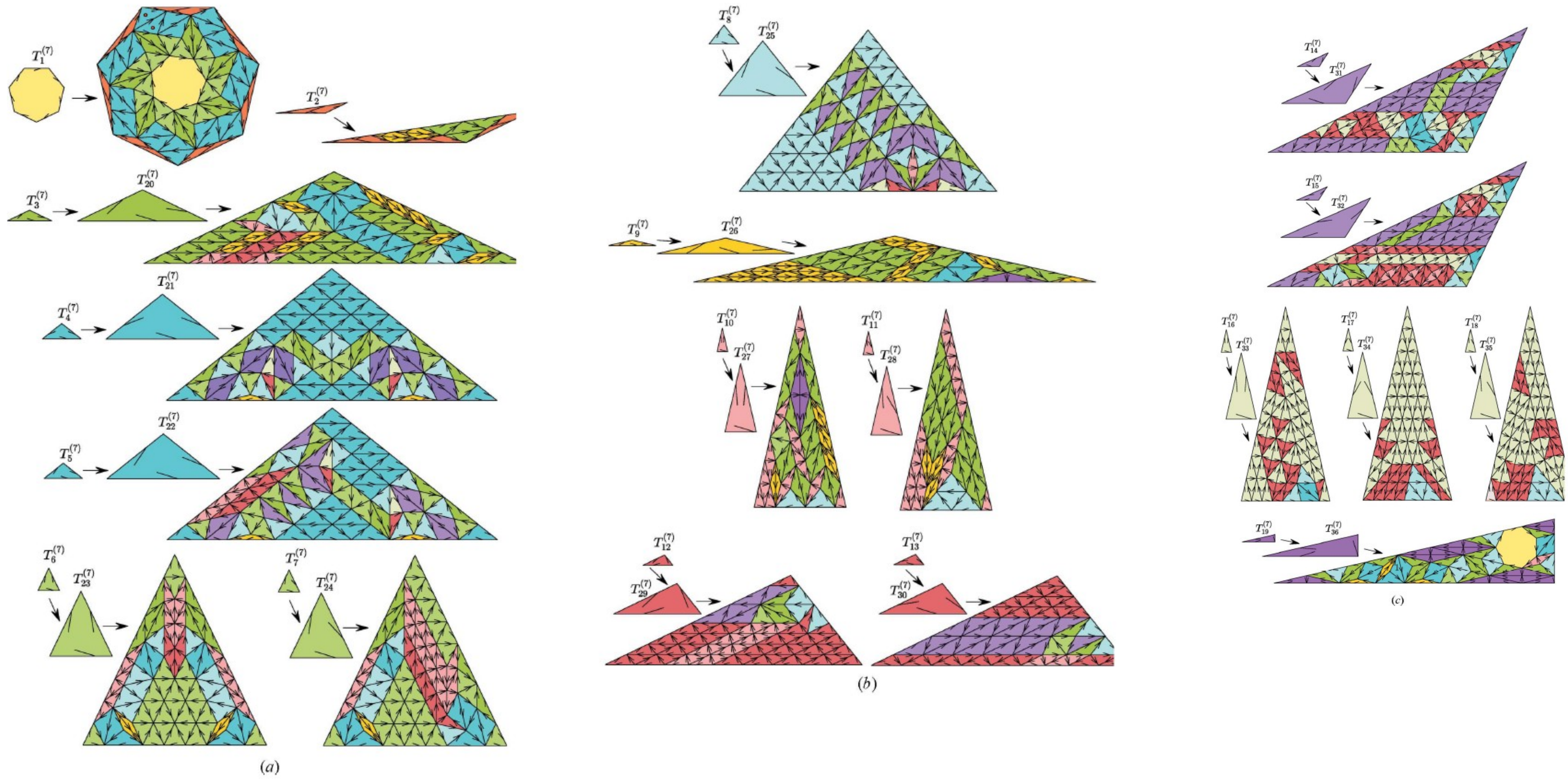




Results from Frettlöh, Say-Awen, De Las Peñas 2017



Results from Say-Awen, De Las Peñas, Frettlöh 2018



Results from Frettlöh, Say-Awen, De Las Peñas 2017

Aperiodic substitution tilings exist with ...

- Individual n-fold cyclical symmetry
- Some proto tiles have irrational interior angles.
  ... that implies: $\arg(\eta) \notin \pi\mathbb{Q}$
- Orientations of the prototiles are dense on the circle
  (DTO = Dense Tile Orientations)

Questions …

- How to derive the minimal inflation multiplier of a cyclotomic aperiodic substitution tiling with dense tile orientations?

- Does cyclotomic aperiodic substitution tilings with dense tile orientations exist with individual n-fold dihedral symmetry?*

  * Subsequent remark:
  The answer is yes. As shown in the examples for n=4 and n=5, individual dihedral symmetry of a CAST with DTO can be ensured by choosing substitution rules with mirror symmetry.

Previous results from Pautze 2017

The Perron–Frobenius eigenvalue λ_1 of the substitution matrix M of a Cyclotomic Aperiodic Substitution Tiling can be written as sum of diagonals of a regular unit n-gon $\mu_{n,k}$ or as sum of pairs of complex conjugated roots of unity $\zeta_{2n}^k + \overline{\zeta_{2n}^k}$ with additional conditions:

$$\lambda_1 = \sum_{k=1}^{\lfloor n/2 \rfloor} c_k \mu_{n,k} \qquad (c_k \in \mathbb{N}_0)$$

Aperiodicity of $\mathcal{T}$ and primitivity of its substitution matrix for $n \geq 4$ are ensured by the following conditions for the coefficients c_k:

$$\max(c_k) \geq 1 \qquad (k \geq 2;\ odd\ n \geq 5)$$

$$\min\left(\left(\max(c_k),\ odd\ k\right), \left(\max(c_k),\ even\ k\right)\right) \geq 1 \qquad (even\ n \geq 4)$$

Previous results from Pautze 2017

The Perron–Frobenius eigenvalue λ_1 of the substitution matrix M of a Cyclotomic Aperiodic Substitution Tiling can be written as sum of diagonals of a regular unit n-gon $\mu_{n,k}$ or as sum of pairs of complex conjugated roots of unity $\zeta_{2n}^k + \overline{\zeta_{2n}^k}$ with additional conditions:

$$\lambda_1 = \ b_0 + \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} b_k \left(\zeta_{2n}^k + \overline{\zeta_{2n}^k} \right) \qquad (b_0, b_k \in \mathbb{N}_0)$$

Aperiodicity of $\mathcal{T}$ and primitivity of its substitution matrix for $n \geq 4$ are ensured by the following conditions for the coefficients c_k:

$$b_k \geq b_{k+2} \qquad (\lfloor (n-1)/2 \rfloor \geq k+2 > k \geq 0)$$

$$\max(b_1, b_2) \geq 1 \qquad (odd\ n \geq 5)$$

$$\min(b_0, b_1) \geq 1 \qquad (even\ n \geq 4)$$

Previous results from Pautze 2017

The smallest possible inflation multipliers of CASTs with $n \geq 4$ are given by:

$$|\eta_{min}| = \left|\zeta_{2n}^{1} + \overline{\zeta_{2n}^{1}}\right| = \mu_{n,2} \qquad (odd\ n \geq 5)$$

$$|\eta_{min}| = \left|1 + \zeta_{2n}^{1}\right| = \sqrt{\mu_{n,2} + 2} \qquad (even\ n \geq 4)$$

Minimal Inflation Multiplier with Irrational Argument

$$\exists z \mid \frac{z}{\overline{z}}\eta \in \mathbb{Z}\left[\zeta_{2n}\right] \leftrightarrow \lambda_1 = \eta \cdot \overline{\eta} \in \mathbb{Z}\left[\zeta_{2n} + \overline{\zeta_{2n}}\right]$$

For simplification we choose $z/\overline{z} = 1$ so that $\eta \in \mathbb{Z}\left[\zeta_{2n}\right]$:

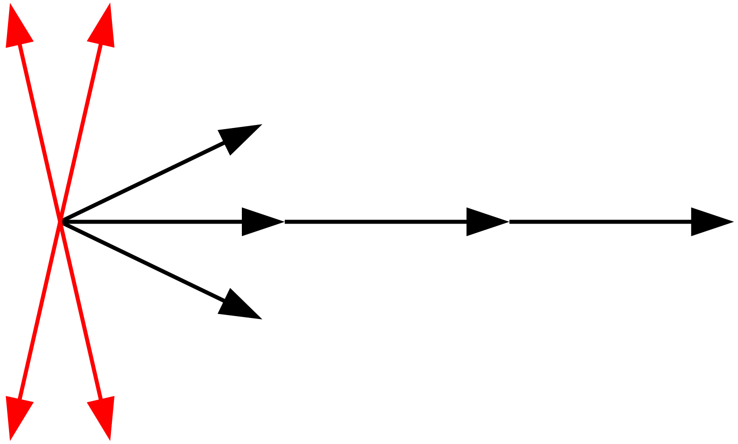
$$\eta = \sum_{k=1}^{l} \zeta_{2n}^{\alpha_k} \in \mathbb{Z}\left[\zeta_{2n}\right] \qquad (\alpha_k \in \mathbb{Z},\ \alpha_l - \alpha_1 < 2n,\ \alpha_k \leq \alpha_{k+1},\ l \geqq 3)$$

Min. Infl. Multiplier with Irrational Argument ... Some Cases

$$odd\ n \geqq 5$$

$$\eta_{min.irr.} = 1 + \zeta_{2n}^{1} + \zeta_{2n}^{(n+1)/2}$$

$$\lambda_{1\ min.irr.} = 3 + \zeta_{2n}^{1} + \overline{\zeta_{2n}^{1}} = 3 + \mu_{n,2}$$



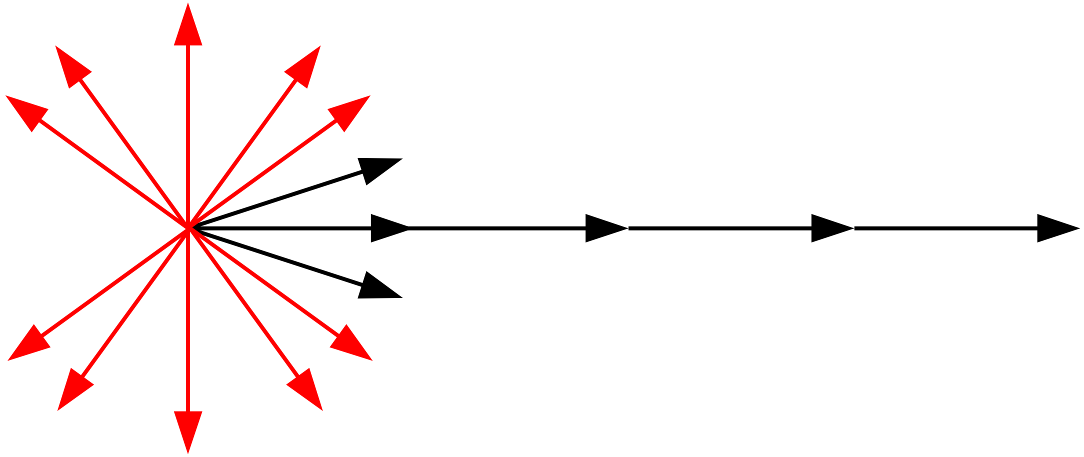
Min. Infl. Multiplier with Irrational Argument ... Some Cases

$$even\ n \geqq 10$$

$$n \bmod 4 = 2$$

$$\eta_{min.irr.} = 1 + \zeta_{2n}^{1} + \zeta_{2n}^{(n+2)/4} + \zeta_{2n}^{(3n+2)/4}$$

$$\lambda_{1\ min.irr.} = 4 + \zeta_{2n}^{1} + \overline{\zeta_{2n}^{1}} = 4 + \mu_{n,2}$$



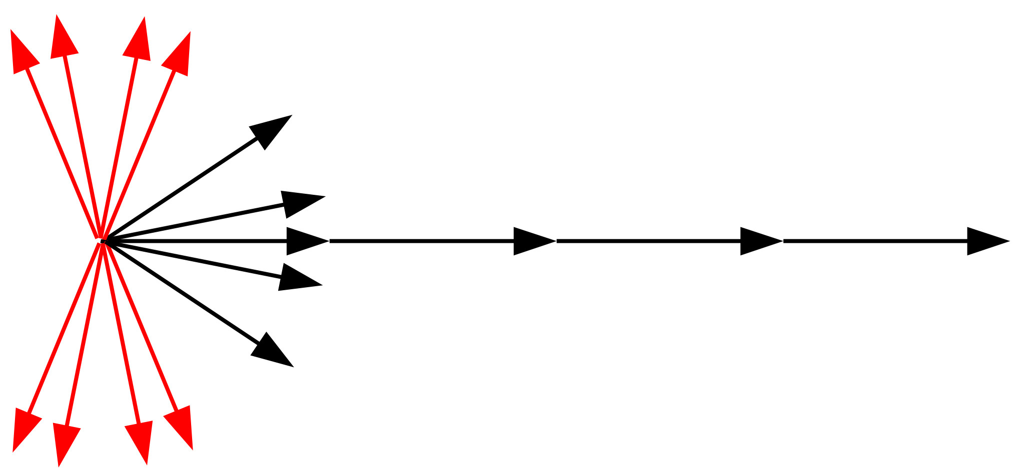
Min. Infl. Multiplier with Irrational Argument ... Some Cases

$$even\ n \geqq 16$$

$$n \bmod 4 = 0$$

$$\eta_{min.irr.} = 1 + \zeta_{2n}^{1} + \zeta_{2n}^{(n-2)/2} + \zeta_{2n}^{(n+4)/2}$$

$$\lambda_{1\ min.irr.} = 4 + \zeta_{2n}^{1} + \overline{\zeta_{2n}^{1}} + \zeta_{2n}^{3} + \overline{\zeta_{2n}^{3}} = 4 + \mu_{n,4}$$



Minimal Inflation Multiplier with Irrational Argument

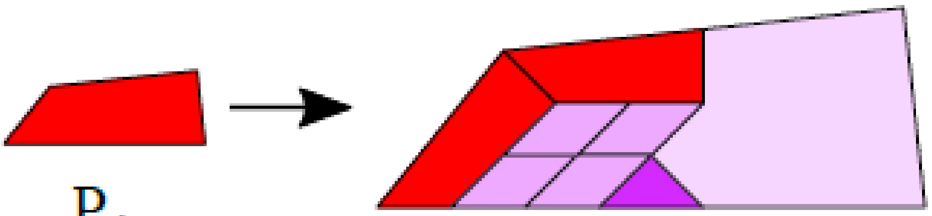
n	Minimal inflation multiplier $\eta_{min.irr.}$	Minimal Perron-Frobenius eigenvalue $\lambda_{1\ min.irr.} = \eta_{min.irr.}\overline{\eta_{min.irr.}}$	Approach
2	$2 + \zeta_4^1$	5	analytic
3	$2 + \zeta_6^1$	3	analytic
4	$1 + 2\zeta_8^1 + \zeta_8^3$	$6 + \zeta_8^1 + \overline{\zeta_8^1} = 6 + \mu_{4,2}$	numeric
6	$1 + \zeta_{12}^1 + \zeta_{12}^3$	$4 + \zeta_{12}^1 + \overline{\zeta_{12}^1} = 4 + \mu_{6,2}$	analytic
8	$1 + \zeta_{16}^1 + \zeta_{16}^4$	$3 + \zeta_{16}^1 + \overline{\zeta_{16}^1} + \zeta_{16}^3 + \overline{\zeta_{16}^3} = 3 + \mu_{8,4}$	analytic
12	$1 + \zeta_{24}^1 + \zeta_{24}^4$	$4 + \zeta_{24}^1 + \overline{\zeta_{24}^1} + \zeta_{24}^3 + \overline{\zeta_{24}^3} = 4 + \mu_{12,4}$	analytic
odd $n \geqq 5$	$1 + \zeta_{2n}^1 + \zeta_{2n}^{(n+1)/2}$	$3 + \zeta_{2n}^1 + \overline{\zeta_{2n}^1} = 3 + \mu_{n,2}$	analytic
5	$1 + \zeta_{10}^1 + \zeta_{10}^3$	$3 + \zeta_{10}^1 + \overline{\zeta_{10}^1} = 3 + \mu_{5,2}$	analytic
7	$1 + \zeta_{14}^1 + \zeta_{14}^4$	$3 + \zeta_{14}^1 + \overline{\zeta_{14}^1} = 3 + \mu_{7,2}$	analytic
9	$1 + \zeta_{18}^1 + \zeta_{18}^5$	$3 + \zeta_{18}^1 + \overline{\zeta_{18}^1} = 3 + \mu_{9,2}$	analytic
	...		
even $n \geqq 10$, $n \bmod 4 = 2$	$1 + \zeta_{2n}^1 + \zeta_{2n}^{(n+2)/4} + \zeta_{2n}^{(3n+2)/4}$	$4 + \zeta_{2n}^1 + \overline{\zeta_{2n}^1} = 4 + \mu_{n,2}$	analytic
10	$1 + \zeta_{20}^1 + \zeta_{20}^3 + \zeta_{20}^8$	$4 + \zeta_{20}^1 + \overline{\zeta_{20}^1} = 4 + \mu_{10,2}$	analytic
14	$1 + \zeta_{28}^1 + \zeta_{28}^4 + \zeta_{28}^{11}$	$4 + \zeta_{28}^1 + \overline{\zeta_{28}^1} = 4 + \mu_{14,2}$	analytic
18	$1 + \zeta_{36}^1 + \zeta_{36}^5 + \zeta_{36}^{14}$	$4 + \zeta_{36}^1 + \overline{\zeta_{36}^1} = 4 + \mu_{18,2}$	analytic
	...		
even $n \geqq 16$, $n \bmod 4 = 0$	$1 + \zeta_{2n}^1 + \zeta_{2n}^{(n-2)/2} + \zeta_{2n}^{(n+4)/2}$	$4 + \zeta_{2n}^1 + \overline{\zeta_{2n}^1} + \zeta_{2n}^3 + \overline{\zeta_{2n}^3} = 4 + \mu_{n,4}$	numeric for $16 \leqq n \leqq 100$, conjecture for $n > 100$
16	$1 + \zeta_{32}^1 + \zeta_{32}^7 + \zeta_{32}^{10}$	$4 + \zeta_{32}^1 + \overline{\zeta_{32}^1} + \zeta_{32}^3 + \overline{\zeta_{32}^3} = 4 + \mu_{16,4}$	numeric
20	$1 + \zeta_{40}^1 + \zeta_{40}^9 + \zeta_{40}^{12}$	$4 + \zeta_{40}^1 + \overline{\zeta_{40}^1} + \zeta_{40}^3 + \overline{\zeta_{40}^3} = 4 + \mu_{20,4}$	numeric
24	$1 + \zeta_{48}^1 + \zeta_{48}^{11} + \zeta_{48}^{14}$	$4 + \zeta_{48}^1 + \overline{\zeta_{48}^1} + \zeta_{48}^3 + \overline{\zeta_{48}^3} = 4 + \mu_{24,4}$	numeric
	...		

Cyclotomic Aperiodic Substitution Tiling with Dense Tile Orientations … Example n = 4 (8th Cyclotomic Field)

$$n = 4$$

$$\eta = \zeta_8^0 + \zeta_8^1 + \zeta_8^1 + \zeta_8^3$$

$$\arg(\eta) \notin \pi\mathbb{Q}$$



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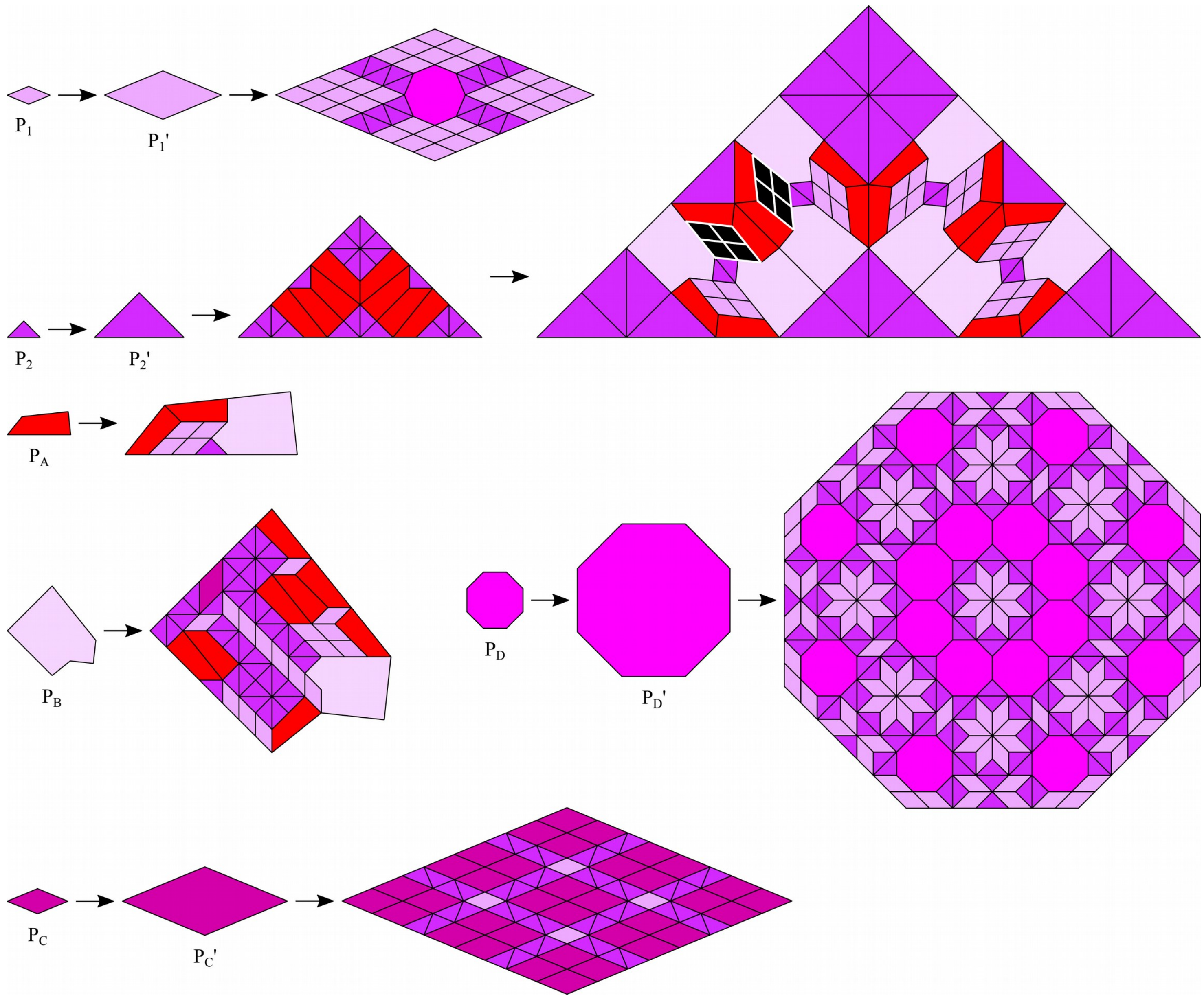
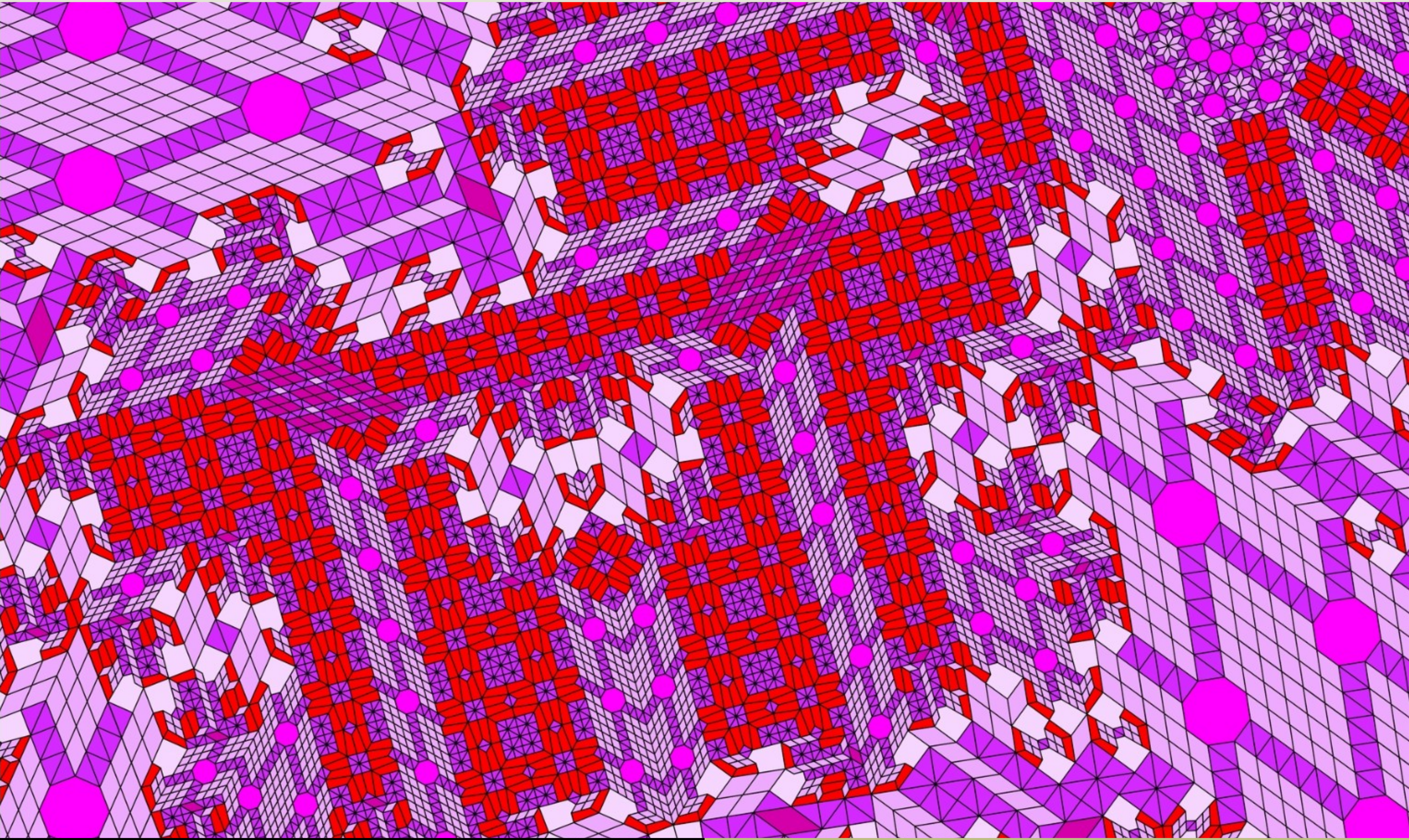
Cyclotomic Aperiodic Substitution Tiling with Dense Tile Orientations … Example n = 4 (8^{th} Cyclotomic Field)



Cyclotomic Aperiodic Substitution Tiling with Dense Tile Orientations ... Example n = 4 (8^{th} Cyclotomic Field)

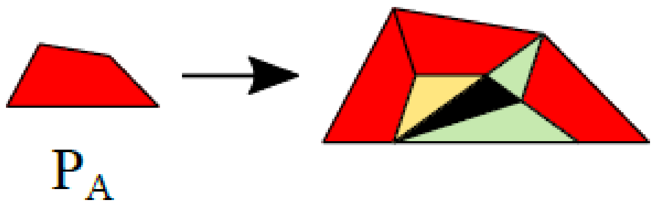


Cyclotomic Aperiodic Substitution Tiling with Dense Tile Orientations ... Example n = 5 (10th Cyclotomic Field)

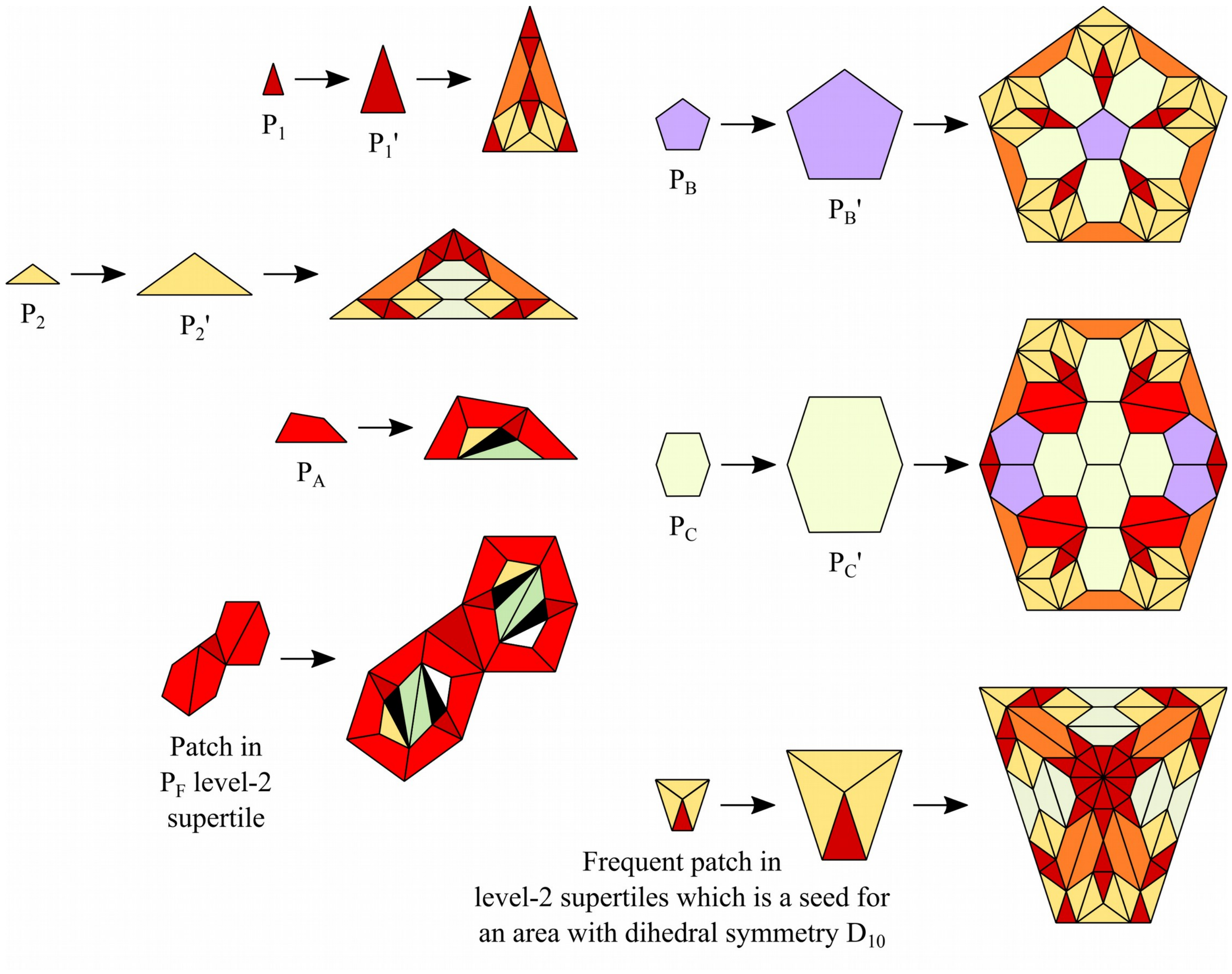
$$n = 5$$

$$\eta = \zeta_{10}^{0} + \zeta_{10}^{1} + \zeta_{10}^{3}$$

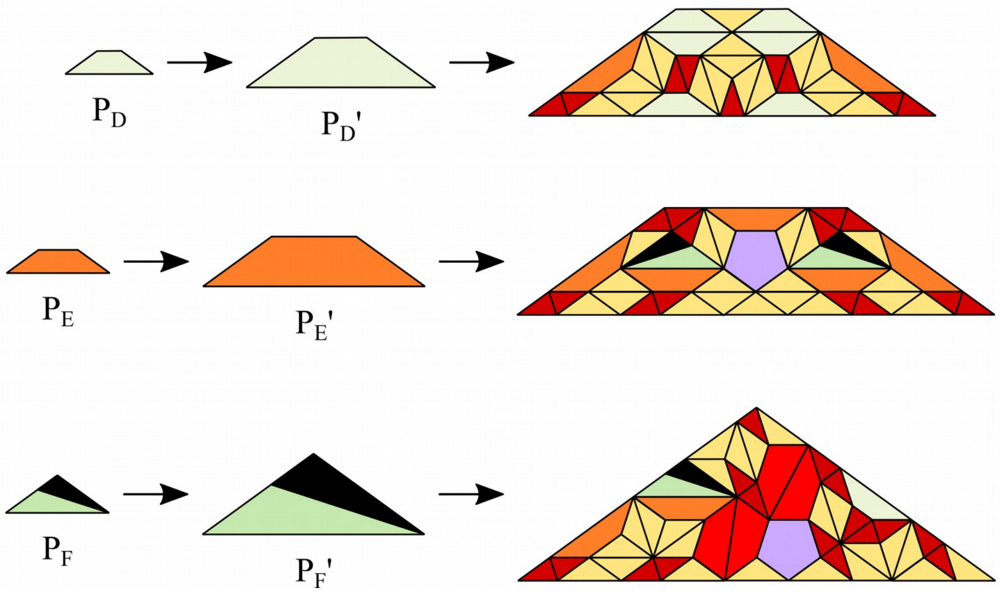
$$\arg(\eta) \notin \pi\mathbb{Q}$$



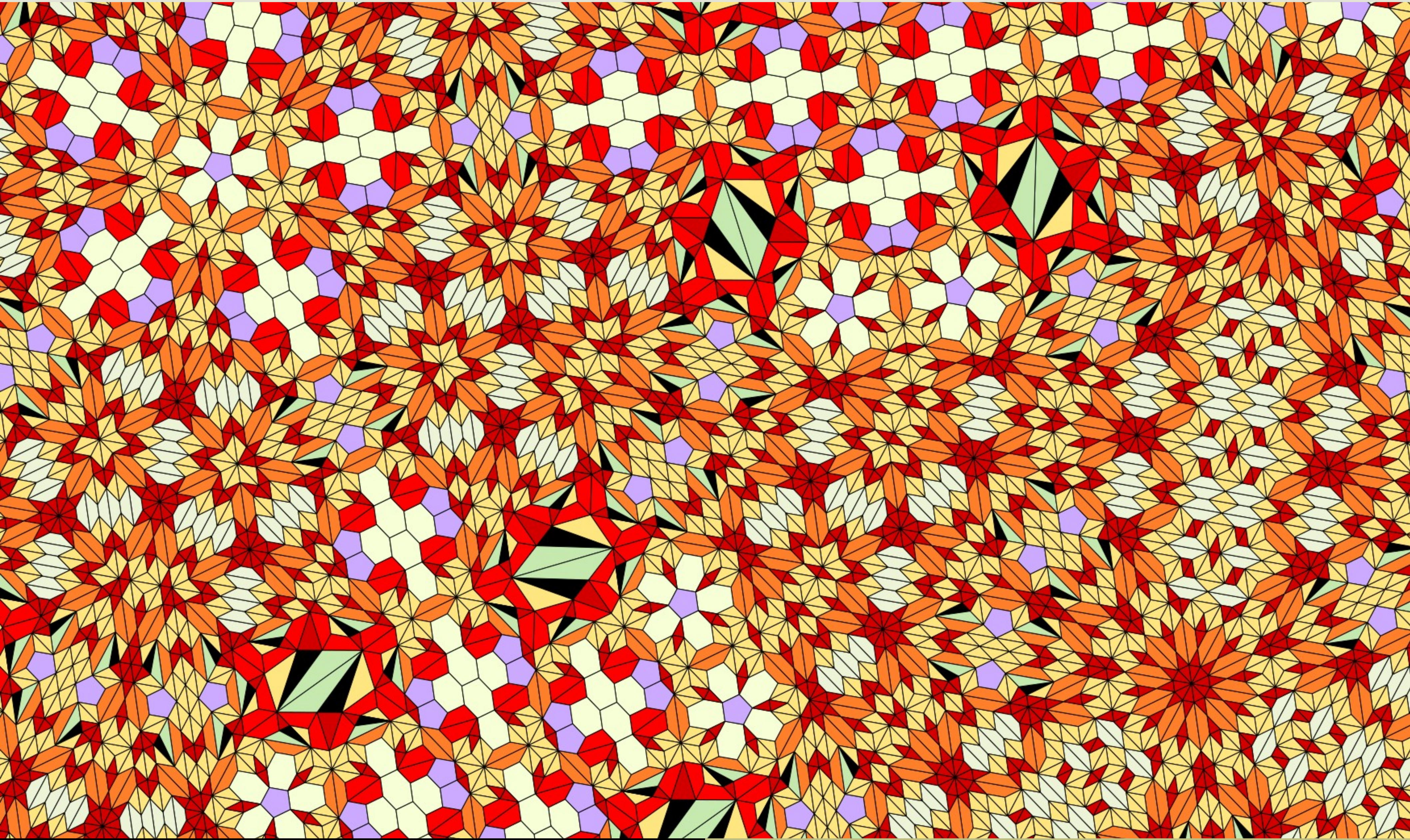
Cyclotomic Aperiodic Substitution Tiling with Dense Tile Orientations … Example n = 5 (10th Cyclotomic Field)



Cyclotomic Aperiodic Substitution Tiling with Dense Tile Orientations ... Example n = 5 (10th Cyclotomic Field)



Cyclotomic Aperiodic Substitution Tiling with Dense Tile Orientations … Example n = 5 (10^{th} Cyclotomic Field)



Cyclotomic Aperiodic Substitution Tilings

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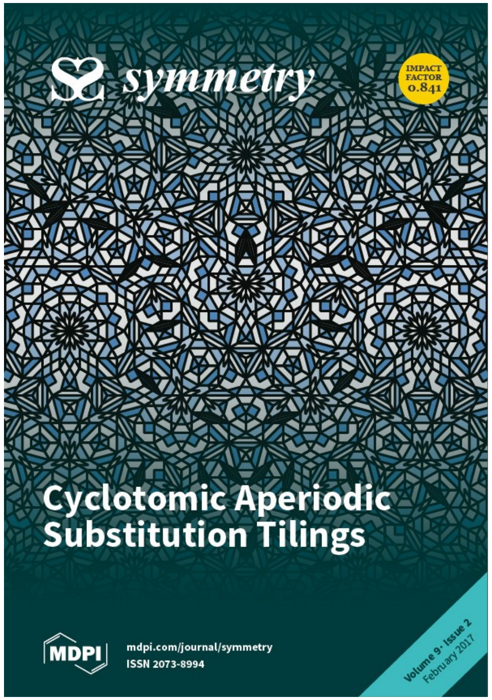
  http://www.mdpi.com/2073-8994/9/2/19



Cyclotomic Aperiodic Substitution Tilings with Dense Tile Orientations

- Preview available at Arxiv.org

  https://arxiv.org/abs/1901.07639

  … revision pending …

Thank you !

Backup / Literature

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